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Bessel's Equation and Bessel Functions: Solution to Schrödinger Equation in a Neumann and Hankel Functions

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Abstract: **In this paper, Bessel's differential equation and Bessel functions were used via the cylindrical coordinates of Laplace equation and the method of Frobenius, the solutions of Schrödinger equation applied to Neumann and Hankel functions were obtained by the application of undetermined coefficients.**

Keywords: **Bessel differential equation, Bessel functions, Hankel functions, Neumann functions and Schrödinger equation.**

I. INTRODUCTION

Bessel functions were studied by Euler, Lagrange and the Daniel Bernoulli. The Bessel functions were first used by Friedrich Wilhelm Bessel to explain the three body motion, with the Bessel function which emerge in the series expansion of planetary perturbation. Bessel function are named for Friedrich Wilhelm Bessel (1784-1846), after all, Daniel Bernoulli is generally attributed with being the first to present the idea of Bessel functions in 1732. He used the function of zero order as a solution to the problem of an oscillating chain hanging at one end. By the year 1764, Leonhard Euler employed Bessel functions of both the integral orders and zero orders in an analysis of vibrations of a stretched membrane, a research that was further developed by Lord Rayleigh in 1878, where he proved that Bessel functions are particular case of Laplace functions (Niedziela, 2008).

Bessel's differential equation arises as a result of determining separable solutions to Laplace's equation and the Helmholtz equation in spherical and cylindrical coordinates. Therefore, Bessel functions are of great important for many problems of wave propagation and static potentials.

Bessel equations were also obtained in solving various classical physics problems. Historically, the equation with v=0 was first experience and solved by Daniel Bernoulli in1732 in his research of the hanging chain problem. Similar equations emerged later in 1770 in the work of Lagrange on astronomical problems. In 1824, the German mathematician and astronomer F.W.Bessel in his research of the problem of elliptic planetary motion come across a special form of Bessel's equation. Influenced by the great work of Fourier that had just emerged in 1822, Bessel conducted an efficient research of Bessel's equation (Asmar, 2005).

The notation Jv, was first used by Hansen (1843) and afterwards by Schlomilch (1857) and later modified to $J(2v)$ by Watson (1922). Subsequent research of Bessel functions included the works of Mathews in 1895, "A treatise on Bessel functions and their applications to physics" written in joint effort with Andrew Gray. It was the first major dissertation on Bessel functions in English and covered topics such as, application of Bessel functions to electricity, hydrodynamics and diffraction. In 1922, Watson first presented his comprehensive analysis of Bessel functions "A dissertation on the theory of Bessel functions".

Frequently, the key to solving such a problems is to identify the form of this equations. Thus, leaving employment of the Bessel functions as solutions. The Frobenius method is used to obtain a Bessel functions which is a solution to Bessel differential equations with variable coefficients. Also we can obtained the Laplace equation in polar coordinates with Bessel equation by using the expression, which is the key equation in mathematical physics, engineering science and basic science and other related fields are common in finding the problems of this equation.

Applications of Bessel functions to the theory of heat conduction, which include dynamical system and heat conduction in spherical or cylindrical objects, which are very large. In the theory of elasticity, the solutions of Bessel functions are efficient for all special problems, which are the solutions of cylindrical or spherical coordinates, and also for various problems relating to the oscillation of plates and equilibrium of plates on an electric foundation, for a series of the questions of theory of shells, for the problems on concentration of the stress near cracks and others. In each of these fields there are many applications of Bessel functions. Different parts of the theory of Bessel functions are extensively used when solving problems of hydrodynamics, acoustics, radio physics, atomic and nuclear physics, quantum physics and so on.

This work has been arranged into various important sections which includes;

. Introduction

. Notation and Assumption

. Result

. Conclusion

II. NOTATIONS AND ASSUMPTIONS

$J_v(x)$	Bessel Function of order v
$\Gamma(x)$	Gamma function
$H_v^{(1)}(x)$	Hankel function of the First kind
$H_v^{(2)}(x)$	Hankel function of the Second kind
∇^2	Laplacian operator
$Y_v(x)$ or $N_v(x)$	Neumann or Weber function
$\hbar$	Planck's constant
$u(x)$	Potential energy
$\Psi(x,t)$	Wave function

III. RESULT

3.1 Solution to Schrödinger equation in a cylindrical functions of the second kind:

Consider the functions J_v and J_{-v} which are two linearly independent solutions of the Bessel's equation, that is;

$$x^2Q'' + xQ' + (x^2 - v^2)Q = 0$$

As representatives of the Neumann or Weber's function. That is,

$$Y_v(x) = N_v(x) = \frac{J_v(x)\cos\pi v - J_{-v}(x)}{\sin\pi v} \quad (1)$$

Which in the Schrödinger equation presents:

$$-\frac{\hbar^2k^2}{2m}\cdot\frac{d^2Y_v}{dx^2} = EY_v(x) \quad (2)$$

Now, we differentiate equation (1) twice and substitute into equation (2), as follows:

$$\frac{dY_v(x)}{dx} = \frac{d}{dx}\left[\frac{J_v(x)\cos \pi v}{\sin \pi v} - \frac{J_{-v}(x)}{\sin \pi v}\right]$$

$$= \frac{\cos \pi v}{\sin \pi v} J'_v(x) - \frac{1}{\sin \pi v} J'_{-v}(x)$$

Again,

$$\frac{d^2 Y_v}{dx^2} = \frac{d}{dx}\left[\frac{\cos \pi v}{\sin \pi v} J'_v(x) - \frac{1}{\sin \pi v} J'_{-v}(x)\right]$$

Implies;

$$\frac{d^2 Y_v}{dx^2} = \frac{\cos \pi v}{\sin \pi v} J''_v(x) - \frac{1}{\sin \pi v} J''_{-v}(x) \qquad (3)$$

Therefore, we substitutes equation (3) into equation (2), we have;

$$-\frac{\hbar^2}{2m}\left[\frac{\cos \pi v}{\sin \pi v} J''_v(x) - \frac{1}{\sin \pi v} J''_{-v}(x)\right] = E\left[\frac{J_v(x)\cos \pi v - J_{-v}(x)}{\sin \pi v}\right]$$

$$\frac{1}{\sin \pi v}\left[\cos \pi v\, J''_v(x) - J''_{-v}(x)\right] = \frac{-2mE}{\hbar^2}\left[\frac{1}{\sin \pi v}\right]\left[J_v(x)\cos \pi v - J_{-v}(x)\right]$$

$$\cos \pi v\, J''_v(x) - J''_{-v}(x) = \frac{-2mE}{\hbar^2}\left[J_v(x)\cos \pi v - J_{-v}(x)\right]$$

By letting $r^2 = \frac{2mE}{\hbar^2}$, we have:

$$\cos \pi v\, J''_v(x) - J''_{-v}(x) = -r^2 \cos \pi v\, J_v(x) + r^2 J_{-v}(x)$$

So that we can obtain;

$$\cos \pi v\, J''_v(x) + r^2 \cos \pi v\, J_v(x) = J''_{-v}(x) + r^2 J_{-v}(x)$$

And we can re-write the above equation as:

$$\cos \pi v\left[J''_v(x) + r^2 J_v(x)\right] = J''_{-v}(x) + r^2 J_{-v}(x) \qquad (4)$$

Therefore, the only way this equation can be equal is when both of them is equal to some constant. That is;

Suppose;

$$J''_{-v}(x) + r^2 J_{-v}(x) = k \qquad (5)$$

$$\cos \pi v\left[J''_v(x) + r^2 J_v(x)\right] = k \qquad (6)$$

To simplify equation (5) and equation (6), we follow the method of undetermined coefficient and obtain the solution as follows;

For equation (5), we have:

$$J_{-v}(x) = c_1 \cos rx + c_2 \sin rx + A_1$$

But, we know that $r = \frac{\sqrt{2mE}}{\hbar}$, therefore;

$$J_{-v}(x) = c_1 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_2 \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_1 \qquad (9)$$

Similarly, for equation (6), we have:

$$J_v(x) = c_3 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_4 \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_2 \qquad (10)$$

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3.2 Solutions to Schrödinger equation in a cylindrical functions of the third kind:

Also, here we are going to apply the Schrödinger equation to cylindrical functions of the third kind (Hankel functions) and obtain the solution of Bessel's equation. The Hankel function of the first and second kind are respectively given by:

$$H_v^{(1)}(x) = J_v(x) + iY_v(x) = i\frac{e^{-v\pi i}J_v(x) - J_{-v}(x)}{\sin \pi v} \tag{11}$$

And

$$H_v^{(2)}(x) = J_v(x) - iY_v(x) = -i\frac{e^{v\pi i}J_v(x) - J_{-v}(x)}{\sin \pi v} \tag{12}$$

Again, on applying equation (11) and equation (12) into the Schrödinger equation as applied to the Hankel function of the first kind, that is;

$$-\frac{\hbar^2}{2m}\cdot\frac{d^2H_v^{(1)}(x)}{dx^2} = EH_v^{(1)}(x) \tag{13}$$

We obtain the solutions as follows:

Now, we differentiate equation (11) twice and substitute into equation (13), we have;

$$H_v^{(1)\prime}(x) = \frac{d}{dx}\left(\frac{ie^{-v\pi i}J_v(x)}{\sin \pi v}\right) - \frac{d}{dx}\left(\frac{iJ_{-v}(x)}{\sin \pi v}\right)$$

$$= \frac{ie^{-v\pi i}}{\sin \pi v}\frac{d}{dx}J_v(x) - \frac{i}{\sin \pi v}\frac{d}{dx}\left(J_{-v}(x)\right)$$

Which implies,

$$H_v^{(1)\prime\prime}(x) = \frac{d}{dx}\left(\frac{ie^{-v\pi i}}{\sin \pi v}\frac{d}{dx}J_v(x)\right) - \frac{d}{dx}\left(\frac{i}{\sin \pi v}\frac{d}{dx}J_{-v}(x)\right)$$

$$= \frac{ie^{-v\pi i}}{\sin \pi v}\frac{d^2}{dx^2}J_v(x) - \frac{i}{\sin \pi v}\frac{d^2}{dx^2}J_{-v}(x)$$

$$= \frac{ie^{-v\pi i}}{\sin \pi v}J_v''(x) - \frac{i}{\sin \pi v}J_{-v}''(x) \tag{14}$$

By replacing equation (14) into equation (13), we have;

$$-\frac{\hbar^2}{2m}\left(\frac{ie^{-v\pi i}}{\sin \pi v}J_v''(x) - \frac{i}{\sin \pi v}J_{-v}''(x)\right) = E\left(\frac{ie^{-v\pi i}}{\sin \pi v}J_v(x) - \frac{i}{\sin \pi v}J_{-v}(x)\right)$$

Letting $a^2 = \frac{2mE}{\hbar^2}$

Implies;

$$\frac{i}{\sin \pi v}J_{-v}''(x) - \frac{ie^{-v\pi i}}{\sin \pi v}J_v''(x) = a^2\left[\frac{ie^{-v\pi i}}{\sin \pi v}J_v(x) - \frac{i}{\sin \pi v}J_{-v}(x)\right]$$

Implies;

$$\frac{i}{\sin \pi v}J_{-v}''(x) - \frac{ie^{-v\pi i}}{\sin \pi v}J_v''(x) = \frac{ia^2e^{-v\pi i}}{\sin \pi v}J_v(x) - \frac{ia^2}{\sin \pi v}J_{-v}(x)$$

$$\frac{i}{\sin \pi v}J_{-v}''(x) + \frac{ia^2}{\sin \pi v}J_{-v}(x) = \frac{ie^{-v\pi i}}{\sin \pi v}J_v''(x) + \frac{ia^2e^{-v\pi i}}{\sin \pi v}J_v(x)$$

Therefore,

$$J_{-v}''(x) + a^2J_{-v}(x) = e^{-v\pi i}\left(J_v''(x) + a^2J_v(x)\right) \tag{15}$$

Again, the only way this equation can be equal is when both of them equal to some constant. That is:

Suppose;

$$J''_{-v}(x) + a^2 J_{-v}(x) = k_1 \tag{16}$$

And

$$e^{-v\pi i}\left(J''_v(x) + a^2 J_v(x)\right) = k_1 \tag{17}$$

Now, by using the method of undetermined coefficients we obtained the solutions as follows;

For equation (16), the solution is;

$$J_{-v}(x) = c_5 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_6 \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_3 \tag{18}$$

And for equation (17), we have;

$$J_v(x) = c_7 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_8 \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_4 \quad (19).$$

Similarly, for the Hankel functions of the second kind in equation (12), we have the solution as follows:

$$J_{-v}(x) = c_9 \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_{10} \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_5 \tag{20}$$

And

$$J_v(x) = c_{11} \cos\left(\frac{\sqrt{2mE}}{\hbar}x\right) + c_{12} \sin\left(\frac{\sqrt{2mE}}{\hbar}x\right) + A_6 \tag{21}$$

Lastly, the $J_v(x)$ and $J_{-v}(x)$ presents in each of the solutions above are two linearly independent solutions of Bessel's differential equation which appears in the cylindrical function of the second and third kind.

IV. CONCLUSION

We have discussed the solution of a free particle (zero potential) time-independent Schrödinger equation as applied to cylindrical function of the second kind (Neumann function) and cylindrical function of the third kind (Hankel function of the first and second kind). It has been find out that, the solution in each case which are presents in the solution of Bessel differential equation are similar. The constants in each of the solution are to be determined via application of boundary conditions. This shows that the Bessel function appeared in many diverse scenarios, more especially in a situation involving cylindrical symmetry.

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